Statistical Inference and the Parallel Transport of Probability

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Based on 2203.08119 (+ 2204.05084, 2205.11543)

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Maximum entropy

Consider a probability density p over some space X satisfying the diffusion process

$$\frac{\partial}{\partial t}p = -\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}Jp\right) + D\frac{\partial^2}{\partial x^2}p$$

with $D = \text{const}$ and $J : X \to \mathbb{R}$ a measurable function.

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Discussed by Jordan, Kinderlehrer, and Otto (1998); Markowich and Villani (2000)

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Stationary solution is the desired Gibbs measure, $\exp\{-\lambda J\}$.

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and therefore as a section s of a line bundle $E \xrightarrow{\pi} X$ with typical fibre $\mathbb{R}_{>0}$. Hence J is also a constraint on the shape of the graph of s.

A quick word on parallel transport

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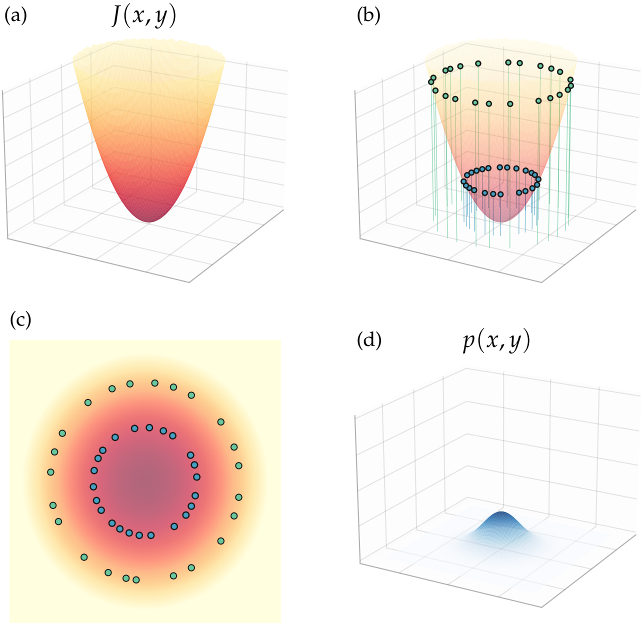
So s should consist of parallel transport lines



Adapted from arXiv:2205.11543. Credit to Brennan Klein.

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$\therefore$ maximising entropy yields the solution to parallel transport with connection valued in $\mathbb{R}$.

A quick sanity check

Stationary solution to constrained maximum entropy:
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Solution to covariant differentiation of a section s in connection $\mathrm{d}J(x)$: $s(x) = \exp\left\{-\lambda \int_k^x \mathrm{d}J(\tilde{x})\right\} = \exp\{-\lambda J(x)\}$

A first idea of gauge theory

Suppose S is a functional on the space of p's, $\Gamma(E)$ and let G be a set. A *gauge* is a quantity g in some Lie group G such that

$$S(\rho(g)\Gamma) = 0$$

for all $g \in G$. A choice of gauge is a choice of one such g. Our $\rho(g)$ is a G-valued transition function t.

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Motivation: changes of frame on TX, where

$$V = V^i \frac{\partial}{\partial x^i} \text{ or } \tilde{V}^i \frac{\partial}{\partial y^i}$$

related by the Jacobian matrix

$$\tilde{V}^j = \frac{\partial y^j}{\partial x^i} V^i$$

A gauge-theoretic interpretation

Define a space of Gibbs measures where a frame therein is given by $\exp\{-J\}$ for some $J : X \to \mathbb{R}$.

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Under $p \mapsto e^{-J'} p$, $J \mapsto J' + J$,

$$\mathcal{L} = -e^{-J} p \ln e^{-J} p - (J' + J) e^{-J} p$$

$$= e^{-J} p(-\ln p - J).$$

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What is the variation? Root of

$$e^{-J}(-\ln p - J)$$

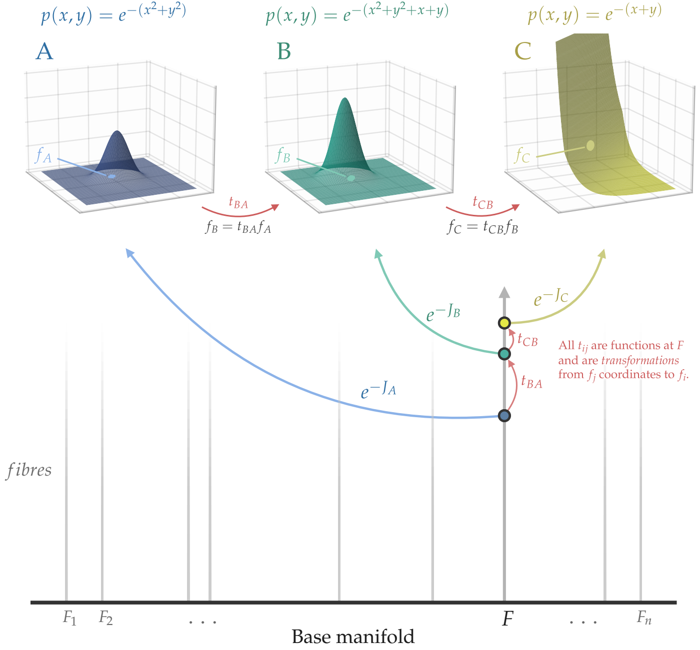
is equal to the root of

$$-\ln p - J.$$

Gibbs frames

Theorem

Let P be the principal bundle associated to E. Since level sets of p are constant with respect to J, there exists some group element $\exp\{J\}$ *for p whose logarithmic derivative is* $\mathrm{d}J$*; moreover, there exists a principal bundle of such Gibbs frames, which is P.*



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